Solving Quadratic Equations by Completing the Square

Focus on...

- solving quadratic equations by completing the square

Rogers Pass gets up to 15 m of snow per year. Because of the steep mountains, over 130 avalanche paths must be monitored during the winter. To keep the Trans-Canada Highway open, the Royal Canadian Artillery uses 105-mm howitzers to create controlled avalanches. The Artillery must aim the howitzer accurately to operate it safely. Suppose that the quadratic function that approximates the trajectory of a shell fired by a howitzer at an angle of 45° is \( h(x) = -\frac{1}{5}x^2 + 2x + \frac{1}{20} \), where \( h \) is the height of the shell and \( x \) is the horizontal distance from the howitzer to where the shell lands, both in kilometres. How can this function be used to determine where to place the howitzer to fire at a specific spot on the mountainside?

Investigate Solving Quadratic Equations by Completing the Square

Materials
- grid paper, graphing calculator, or computer with graphing software

Sometimes factoring quadratic equations is not practical. In Chapter 3, you learned how to complete the square to analyse and graph quadratic functions. You can complete the square to help solve quadratic equations such as \(-\frac{1}{5}x^2 + 2x + \frac{1}{20} = 0\).

1. Graph the function \( f(x) = -\frac{1}{5}x^2 + 2x + \frac{1}{20} \).
2. What are the \( x \)-intercepts of the graph? How accurate are your answers? Why might it be important to determine more accurate zeros for the function?
3. a) Rewrite the function in the form \( h(x) = a(x - p)^2 + q \) by completing the square.
   b) Set \( h(x) \) equal to zero. Solve for \( x \). Express your answers as exact values.

Reflect and Respond

4. What are the two roots of the quadratic equation for projectile motion, \( 0 = -\frac{1}{5}x^2 + 2x + \frac{1}{20} \)? What do the roots represent in this situation?
5. To initiate an avalanche, the howitzer crew must aim the shell up the slope of the mountain. The shot from the howitzer lands 750 m above where the howitzer is located. How could the crew determine the horizontal distance from the point of impact at which the howitzer must be located? Explain your reasoning. Calculate the horizontal distances involved in this scenario. Include a sketch of the path of the projectile.

6. At which horizontal distance from the point of impact would you locate the howitzer if you were in charge of setting off a controlled avalanche? Explain your reasoning.

Did You Know?

Parks Canada operates the world’s largest mobile avalanche control program to keep the Trans-Canada Highway and the Canadian Pacific Railway operating through Rogers Pass.

Link the Ideas

You can solve quadratic equations of the form \( ax^2 + bx + c = 0 \), where \( b = 0 \), or of the form \( a(x - p)^2 + q = 0 \), where \( a \neq 0 \), that have real-number solutions by isolating the squared term and taking the square root of both sides. The square root of a positive real number can be positive or negative, so there are two possible solutions to these equations.

To solve \( x^2 = 9 \), take the square root of both sides.

\[
\pm \sqrt{x^2} = \pm \sqrt{9}
\]

\[
x = \pm 3
\]

Read \( \pm \) as "plus or minus."

3 is a solution to the equation because \( (3)(3) = 9 \).

\(-3\) is a solution to the equation because \( (-3)(-3) = 9 \).

To solve \( (x - 1)^2 - 49 = 0 \), isolate the squared term and take the square root of both sides.

\[
(x - 1)^2 - 49 = 0
\]

\[
(x - 1)^2 = 49
\]

\[
x - 1 = \pm 7
\]

\[
x = 1 \pm 7
\]

\[
x = 1 + 7 \quad \text{or} \quad x = 1 - 7
\]

\[
x = 8 \quad \text{or} \quad x = -6
\]

Check:

Substitute \( x = 8 \) and \( x = -6 \) into the original equation.

<table>
<thead>
<tr>
<th>Left Side</th>
<th>Right Side</th>
</tr>
</thead>
<tbody>
<tr>
<td>((x - 1)^2 - 49)</td>
<td>0</td>
</tr>
<tr>
<td>((8 - 1)^2 - 49)</td>
<td>0</td>
</tr>
<tr>
<td>(7^2 - 49)</td>
<td>(= 49 - 49)</td>
</tr>
<tr>
<td>(49 - 49)</td>
<td>(= 0)</td>
</tr>
</tbody>
</table>

Left Side = Right Side

Left Side = Right Side

Both solutions are correct. The roots are 8 and -6.

Did You Know?

Around 830 c.e., Abu Ja’far Muhammad ibn Musa al-Khwarizmi wrote *Hisab al-jabr w’al-muqabala*. The word *al-jabr* from this title is the basis of the word we use today, *algebra*. In his book, al-Khwarizmi describes how to solve a quadratic equation by completing the square.

Web Link

To learn more about al-Khwarizmi, go to [www.mhrprecalc11.ca](http://www.mhrprecalc11.ca) and follow the links.
Many quadratic equations cannot be solved by factoring. In addition, graphing the corresponding functions may not result in exact solutions. You can write a quadratic function expressed in standard form, \( y = ax^2 + bx + c \), in vertex form, \( y = a(x - p)^2 + q \), by completing the square. You can also use the process of completing the square to determine exact solutions to quadratic equations.

**Example 1**

**Write and Solve a Quadratic Equation by Taking the Square Root**

A wide-screen television has a diagonal measure of 42 in. The width of the screen is 16 in. more than the height. Determine the dimensions of the screen, to the nearest tenth of an inch.

**Solution**

Draw a diagram. Let \( h \) represent the height of the screen. Then, \( h + 16 \) represents the width of the screen.

\[
\begin{align*}
42 \text{ in.} & \\
\hline
\hline
h & \\
\hline
h + 16 & \\
\hline
\end{align*}
\]

Use the Pythagorean Theorem.

\[
\begin{align*}
h^2 + (h + 16)^2 &= 42^2 \\
h^2 + h^2 + 32h + 256 &= 1764 \\
2h^2 + 32h &= 1508 \\
2h^2 + 32h &= 1508 \\
\frac{h + 16}{h} &= 754 \\
\frac{h^2 + 16h + 64}{h} &= 754 + 64 \\
(h + 8)^2 &= 818 \\
h + 8 &= \pm\sqrt{818} \\
h &= -8 \pm \sqrt{818}
\end{align*}
\]

Isolate the variable terms on the left side.
Add the square of half the coefficient of \( h \) to both sides.
Factor the perfect square trinomial on the left side.
Take the square root of both sides.

\[
\begin{align*}
h = -8 + \sqrt{818} & \quad \text{or} \quad h = -8 - \sqrt{818} \\
h & \approx 20.6 & \quad h \approx -36.6
\end{align*}
\]

Since the height of the screen cannot be negative, \( h = -36.6 \) is an extraneous root.

Thus, the height of the screen is approximately 20.6 in., and the width of the screen is approximately 20.6 + 16 or 36.6 in..

Hence, the dimensions of a 42-in. television are approximately 20.6 in. by 36.6 in..

Check:

\( 20.6^2 + 36.6^2 = 1763.92 \), and \( \sqrt{1763.92} \) is approximately 42, the diagonal of the television, in inches.
Your Turn

The circular Canadian two-dollar coin consists of an aluminum and bronze core and a nickel outer ring. If the radius of the inner core is 0.84 cm and the area of the circular face of the coin is $1.96\pi$ cm$^2$, what is the width of the outer ring?

Example 2

Solve a Quadratic Equation by Completing the Square When $a = 1$

Solve $x^2 - 21 = -10x$ by completing the square. Express your answers to the nearest tenth.

Solution

\[
\begin{align*}
   x^2 - 21 &= -10x \\
   x^2 + 10x &= 21 \\
   x^2 + 10x + 25 &= 21 + 25 \\
   (x + 5)^2 &= 46 \\
   x + 5 &= \pm\sqrt{46}
\end{align*}
\]

Solve for $x$.

\[
\begin{align*}
   x + 5 &= \sqrt{46} & \text{or} & & x + 5 &= -\sqrt{46} \\
   x &= -5 + \sqrt{46} & & x &= -5 - \sqrt{46} \\
   x &= 1.7823... & & x &= -11.7823...
\end{align*}
\]

The exact roots are $-5 + \sqrt{46}$ and $-5 - \sqrt{46}$.
The roots are 1.8 and $-11.8$, to the nearest tenth.

You can also see the solutions to this equation graphically as the x-intercepts of the graph of the function $f(x) = x^2 + 10x - 21$.

These occur at approximately $(-11.8, 0)$ and $(1.8, 0)$ and have values of $-11.8$ and 1.8, respectively.

Your Turn

Solve $p^2 - 4p = 11$ by completing the square. Express your answers to the nearest tenth.
Example 3

Solve a Quadratic Equation by Completing the Square When \( a \neq 1 \)

Determine the roots of \(-2x^2 - 3x + 7 = 0\), to the nearest hundredth. Then, use technology to verify your answers.

Solution

\[-2x^2 - 3x + 7 = 0\]

\[x^2 + \frac{3}{2}x - \frac{7}{2} = 0\]

Divide both sides by a factor of \(-2\).

\[x^2 + \frac{3}{2}x = \frac{7}{2}\]

Isolate the variable terms on the left side.

\[x^2 + \frac{3}{2}x + \frac{9}{16} = \frac{7}{2} + \frac{9}{16}\]

Why is \(\frac{9}{16}\) added to both sides?

\[\left(x + \frac{3}{4}\right)^2 = \frac{65}{16}\]

Solve for \(x\).

\[x + \frac{3}{4} = \pm \sqrt{\frac{65}{16}}\]

\[x = -\frac{3}{4} \pm \frac{\sqrt{65}}{4}\]

The exact roots are \(-\frac{3 + \sqrt{65}}{4}\) and \(-\frac{3 - \sqrt{65}}{4}\).

The roots are 1.27 and -2.77, to the nearest hundredth.

Your Turn

Determine the roots of the equation \(-2x^2 - 5x + 2 = 0\), to the nearest hundredth. Verify your solutions using technology.
Example 4

Apply Completing the Square

A defender kicks a soccer ball away from her own goal. The path of the kicked soccer ball can be approximated by the quadratic function $h(x) = -0.06x^2 + 3.168x - 35.34$, where $x$ is the horizontal distance travelled, in metres, from the goal line and $h$ is the height, in metres.

a) You can determine the distance the soccer ball is from the goal line by solving the corresponding equation,

$-0.06x^2 + 3.168x - 35.34 = 0$. How far is the soccer ball from the goal line when it is kicked? Express your answer to the nearest tenth of a metre.

b) How far does the soccer ball travel before it hits the ground?

Solution

a) Solve the equation $-0.06x^2 + 3.168x - 35.34 = 0$ by completing the square.

$-0.06x^2 + 3.168x - 35.34 = 0$

$\frac{x^2}{52.8} - \frac{589}{52.8} = 0$

$x^2 - 52.8x = 589$

Divide both sides by a common factor of $-0.06$.

$x^2 - 52.8x + \left(\frac{52.8}{2}\right)^2 = -589 + \left(\frac{52.8}{2}\right)^2$

$x^2 - 52.8x + 696.96 = -589 + 696.96$

$(x - 26.4)^2 = 107.96$

$x - 26.4 = \pm\sqrt{107.96}$

Complete the square on the left side.

$x - 26.4 = \pm\sqrt{107.96}$

Isolate the variable terms on the left side.

$x = 26.4 + \sqrt{107.96}$

$x = 26.4 - \sqrt{107.96}$

Take the square root of both sides.

$x = 26.4 + \sqrt{107.96}$

$x = 26.4 - \sqrt{107.96}$

$x = 36.7903...$

$x = 16.0096...$

Solve for $x$.

The roots of the equation are approximately 36.8 and 16.0.

The ball is kicked approximately 16.0 m from the goal line.

b) From part a), the soccer ball is kicked approximately 16.0 m from the goal line. The ball lands approximately 36.8 m from the goal line. Therefore, the soccer ball travels 36.8 - 16.0, or 20.8 m, before it hits the ground.

Your Turn

How far does the soccer ball in Example 4 travel if the function that models its trajectory is $h(x) = -0.016x^2 + 1.152x - 15.2$?
Key Ideas

- Completing the square is the process of rewriting a quadratic polynomial from the standard form, $ax^2 + bx + c$, to the vertex form, $a(x - p)^2 + q$.

- You can use completing the square to determine the roots of a quadratic equation in standard form.

  For example,
  
  
  \[
  2x^2 - 4x - 2 = 0 \\
  x^2 - 2x - 1 = 0 \\
  x^2 - 2x = 1 \\
  x^2 - 2x + 1 = 1 + 1 \\
  (x - 1)^2 = 2 \\
  x - 1 = \pm\sqrt{2} \\
  
  x - 1 = \sqrt{2} \quad \text{or} \quad x - 1 = -\sqrt{2} \\
  x = 1 + \sqrt{2} \quad \text{or} \quad x = 1 - \sqrt{2} \\
  x \approx 2.41 \quad \text{or} \quad x \approx -0.41
  \]

  Divide both sides by a common factor of 2.
  Isolate the variable terms on the left side.
  Complete the square on the left side.
  Take the square root of both sides.
  Solve for $x$.

- Express roots of quadratic equations as exact roots or as decimal approximations.

Check Your Understanding

Practise

1. What value of $c$ makes each expression a perfect square?
   a) $x^2 + x + c$
   b) $x^2 - 5x + c$
   c) $x^2 - 0.5x + c$
   d) $x^2 + 0.2x + c$
   e) $x^2 + 15x + c$
   f) $x^2 - 9x + c$

2. Complete the square to write each quadratic equation in the form $(x + p)^2 = q$.
   a) $2x^2 + 8x + 4 = 0$
   b) $-3x^2 - 12x + 5 = 0$
   c) $\frac{1}{2}x^2 - 3x + 5 = 0$

3. Write each equation in the form $a(x - p)^2 + q = 0$.
   a) $x^2 - 12x + 9 = 0$
   b) $5x^2 - 20x - 1 = 0$
   c) $-2x^2 + x - 1 = 0$
   d) $0.5x^2 + 2.1x + 3.6 = 0$
   e) $-1.2x^2 - 5.1x - 7.4 = 0$
   f) $\frac{1}{2}x^2 + 3x - 6 = 0$

4. Solve each quadratic equation. Express your answers as exact roots.
   a) $x^2 = 64$
   b) $2s^2 - 8 = 0$
   c) $\frac{1}{3}t^2 - 1 = 11$
   d) $-y^2 + 5 = -6$
5. Solve. Express your answers as exact roots.
   a) \((x - 3)^2 = 4\)
   b) \((x + 2)^2 = 9\)
   c) \(\left( d + \frac{1}{2} \right)^2 = 1\)
   d) \(\left( h - \frac{3}{4} \right)^2 = \frac{7}{16}\)
   e) \((s + 6)^2 = \frac{3}{4}\)
   f) \((x + 4)^2 = 18\)

6. Solve each quadratic equation by completing the square. Express your answers as exact roots.
   a) \(x^2 + 10x + 4 = 0\)
   b) \(x^2 - 8x + 13 = 0\)
   c) \(3x^2 + 6x + 1 = 0\)
   d) \(-2x^2 + 4x + 3 = 0\)
   e) \(-0.1x^2 - 0.6x + 0.4 = 0\)
   f) \(0.5x^2 - 4x - 6 = 0\)

7. Solve each quadratic equation by completing the square. Express your answers to the nearest tenth.
   a) \(x^2 - 8x - 4 = 0\)
   b) \(-3x^2 + 4x + 5 = 0\)
   c) \(\frac{1}{2}x^2 - 6x - 5 = 0\)
   d) \(0.2x^2 + 0.12x - 11 = 0\)
   e) \(-\frac{2}{3}x^2 - x + 2 = 0\)
   f) \(\frac{3}{4}x^2 + 6x + 1 = 0\)

Apply

8. Dinah's rectangular dog kennel measures 4 ft by 10 ft. She plans to double the area of the kennel by extending each side by an equal amount.
   a) Sketch and label a diagram to represent this situation.
   b) Write the equation to model the new area.
   c) What are the dimensions of the new dog kennel, to the nearest tenth of a foot?

9. Evan passes a flying disc to a teammate during a competition at the Flatland Ultimate and Cups Tournament in Winnipeg. The flying disc follows the path \(h(d) = -0.02d^2 + 0.4d + 1\), where \(h\) is the height, in metres, and \(d\) is the horizontal distance, in metres, that the flying disc has travelled from the thrower. If no one catches the flying disc, the height of the disc above the ground when it lands can be modelled by \(h(d) = 0\).
   a) What quadratic equation can you use to determine how far the disc will travel if no one catches it?
   b) How far will the disc travel if no one catches it? Express your answer to the nearest tenth of a metre.

Did You Know?

Each August, teams compete in the Canadian Ultimate Championships for the national title in five different divisions: juniors, masters, mixed, open, and women's. This tournament also determines who will represent Canada at the world championships.

10. A model rocket is launched from a platform. Its trajectory can be approximated by the function \(h(d) = -0.01d^2 + 2d + 1\), where \(h\) is the height, in metres, of the rocket and \(d\) is the horizontal distance, in metres, the rocket travels. How far does the rocket land from its launch position? Express your answer to the nearest tenth of a metre.
11. Brian is placing a photograph behind a 12-in. by 12-in. piece of matting. He positions the photograph so the matting is twice as wide at the top and bottom as it is at the sides. The visible area of the photograph is 54 sq. in. What are the dimensions of the photograph?

12. The path of debris from fireworks when the wind is about 25 km/h can be modelled by the quadratic function \( h(x) = -0.04x^2 + 2x + 8 \), where \( h \) is the height and \( x \) is the horizontal distance travelled, both measured in metres. How far away from the launch site will the debris land? Express your answer to the nearest tenth of a metre.

**Extend**

13. Write a quadratic equation with the given roots.
   a) \( \sqrt{7} \) and \(-\sqrt{7}\)
   b) \( 1 + \sqrt{3} \) and \( 1 - \sqrt{3} \)
   c) \( \frac{5 + \sqrt{11}}{2} \) and \( \frac{5 - \sqrt{11}}{2} \)

14. Solve each equation for \( x \) by completing the square.
   a) \( x^2 + 2x = k \)
   b) \( kx^2 - 2x = k \)
   c) \( x^2 = kx + 1 \)

15. Determine the roots of \( ax^2 + bx + c = 0 \) by completing the square. Can you use this result to solve any quadratic equation? Explain.

16. The sum of the first \( n \) terms, \( S_n \), of an arithmetic series can be found using the formula \( S_n = \frac{n}{2}[2t_1 + (n - 1)d] \), where \( t_1 \) is the first term and \( d \) is the common difference.
   a) The sum of the first \( n \) terms in the arithmetic series 
      \( 6 + 10 + 14 + \cdots \) is 3870. Determine the value of \( n \).
   b) The sum of the first \( n \) consecutive natural numbers is 780. Determine the value of \( n \).

17. A machinist in a fabrication shop needs to bend a metal rod at an angle of 60° at a point 4 m from one end of the rod so that the ends of the rod are 12 m apart, as shown.

   a) Using the cosine law, write a quadratic equation to represent this situation.
   b) Solve the quadratic equation. How long is the rod, to the nearest tenth of a metre?

**Create Connections**

18. The solution to \( x^2 = 9 \) is \( x = \pm 3 \). The solution to the equation \( x = \sqrt{9} \) is \( x = 3 \). Explain why the solutions to the two equations are different.