Factoring Quadratic Equations

Focus on...
- factoring a variety of quadratic expressions
- factoring to solve quadratic equations
- solving problems involving quadratic equations

Football, soccer, basketball, and volleyball are just a few examples of sports that involve throwing, kicking, or striking a ball. Each time a ball or projectile sails through the air, it follows a trajectory that can be modelled with a quadratic function.

Each of these sports is played on a rectangular playing area. The playing area for each sport can be modelled by a quadratic equation.

Investigate Solving Quadratic Equations by Factoring

Materials
- grid paper or graphing technology

1. For women’s indoor competition, the length of the volleyball court is twice its width. If \( x \) represents the width, then \( 2x \) represents the length. The area of the court is 162 m\(^2\).
   a) Write a quadratic equation in standard form, \( A(x) = 0 \), to represent the area of the court.
   b) Graph the corresponding quadratic function. How many \( x \)-intercepts are there? What are they?
   c) From your graph, what are the roots of the quadratic equation you wrote in part a)? How do you know these are the roots of the equation?
   d) In this context, are all the roots acceptable? Explain.

2. a) Factor the left side of the quadratic equation you wrote in step 1a).
   b) Graph the corresponding quadratic function in factored form. Compare your graph to the graph you created in step 1b).
   c) How is the factored form of the equation related to the \( x \)-intercepts of the graph?
   d) How can you use the \( x \)-intercepts of a graph, \( x = r \) and \( x = s \), to write a quadratic equation in standard form?
3. For men’s sitting volleyball, a Paralympic sport, the length of the court is 4 m more than the width. The area of the court is 60 m².
   a) If \( x \) represents the width, write a quadratic equation in standard form to represent the area of the court.
   b) Graph the corresponding quadratic function. How many \( x \)-intercepts are there? What are they?

4. a) Use the \( x \)-intercepts, \( x = r \) and \( x = s \), of your graph in step 3 to write the quadratic equation \((x - r)(x - s) = 0\).
   b) Graph the corresponding quadratic function. Compare your graph to the graph you created in step 3.

**Reflect and Respond**

5. How does the factored form of a quadratic equation relate to the \( x \)-intercepts of the graph, the zeros of the quadratic function, and the roots of the equation?

6. Describe how you can factor the quadratic equation \( 0 = x^2 - 5x - 6 \) to find the roots.

7. The roots of a quadratic equation are 3 and -5. What is a possible equation?

**Did You Know?**

Volleyball is the world’s number two participation sport. Which sport do you think is number one?

**Link the Ideas**

**Factoring Quadratic Expressions**

To factor a trinomial of the form \( ax^2 + bx + c \), where \( a \neq 0 \), first factor out common factors, if possible.

For example,
\[
4x^2 - 2x - 12 = 2(2x^2 - x - 6) = 2(2x^2 - 4x + 3x - 6) = 2[2x(x - 2) + 3(x - 2)] = 2(x - 2)(2x + 3)
\]

You can factor perfect square trinomials of the forms \((ax)^2 + 2abx + b^2\) and \((ax)^2 - 2abx + b^2\) into \((ax + b)^2\) and \((ax - b)^2\), respectively.

For example,
\[
4x^2 + 12x + 9 = (2x + 3)(2x + 3) \quad 9x^2 - 24x + 16 = (3x - 4)(3x - 4) = (3x - 4)^2
\]

You can factor a difference of squares, \((ax)^2 - (by)^2\), into \((ax - by)(ax + by)\).

For example,
\[
\frac{4}{9}x^2 - 16y^2 = \left(\frac{2}{3}x - 4y\right)\left(\frac{2}{3}x + 4y\right)
\]
**Factoring Polynomials Having a Quadratic Pattern**

You can extend the patterns established for factoring trinomials and a difference of squares to factor polynomials in quadratic form. You can factor a polynomial of the form \( a(P)^2 + b(P) + c \), where \( P \) is any expression, as follows:

- Treat the expression \( P \) as a single variable, say \( r \), by letting \( r = P \).
- Factor as you have done before.
- Replace the substituted variable \( r \) with the expression \( P \).
- Simplify the expression.

For example, in \( 3(x + 2)^2 - 13(x + 2) + 12 \), substitute \( r \) for \( x + 2 \) and factor the resulting expression, \( 3r^2 - 13r + 12 \).

\[
3r^2 - 13r + 12 = (3r - 4)(r - 3)
\]

Once the expression in \( r \) is factored, you can substitute \( x + 2 \) back in for \( r \).

The resulting expression is

\[
(3(x + 2) - 4)(x + 2 - 3) = (3x + 6 - 4)(x - 1)
= (3x + 2)(x - 1)
\]

You can factor a polynomial in the form of a difference of squares, as \( P^2 - Q^2 = (P - Q)(P + Q) \) where \( P \) and \( Q \) are any expressions.

For example,

\[
(3x + 1)^2 - (2x - 3)^2 = [(3x + 1) - (2x - 3)][(3x + 1) + (2x - 3)]
= (3x + 1 - 2x + 3)(3x + 1 + 2x - 3)
= (x + 4)(5x - 2)
\]

---

**Example 1**

**Factor Quadratic Expressions**

Factor.

a) \( 2x^2 - 2x - 12 \)

b) \( \frac{1}{4}x^2 - x - 3 \)

c) \( 9x^2 - 0.64y^2 \)

**Solution**

a) **Method 1: Remove the Common Factor First**

Factor out the common factor of 2.

\( 2x^2 - 2x - 12 = 2(x^2 - x - 6) \)

Find two integers with a product of \(-6\) and a sum of \(-1\).

<table>
<thead>
<tr>
<th>Factors of (-6)</th>
<th>Product</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, (-6)</td>
<td>(-6)</td>
<td>(-5)</td>
</tr>
<tr>
<td>2, (-3)</td>
<td>(-6)</td>
<td>(-1)</td>
</tr>
<tr>
<td>3, (-2)</td>
<td>(-6)</td>
<td>(1)</td>
</tr>
<tr>
<td>6, (-1)</td>
<td>(-6)</td>
<td>(5)</td>
</tr>
</tbody>
</table>
The factors are $x + 2$ and $x - 3$.

$2x^2 - 2x - 12 = 2(x^2 - x - 6)$

$= 2(x + 2)(x - 3)$

**Method 2: Factor the Trinomial First by Grouping**

To factor $2x^2 - 2x - 12$, find two integers with

- a product of $(2)(-12) = -24$
- a sum of $-2$

The two integers are $-6$ and $4$.

Write $-2x$ as the sum $-6x + 4x$.

Then, factor by grouping.

$2x^2 - 2x - 12 = 2x^2 - 6x + 4x - 12$

$= 2(x - 3) + 4(x - 3)$

$= (2x + 4)(x - 3)$

$= 2(x + 2)(x - 3)$

**b)** Factor out the common factor of $\frac{1}{4}$ first.

$\frac{1}{4}x^2 - x - 3 = \frac{1}{4}(x^2 - 4x - 12)$

$= \frac{1}{4}(x + 2)(x - 6)$

**c)** The binomial $9x^2 - 0.64y^2$ is a difference of squares.

The first term is a perfect square: $(3x)^2$

The second term is a perfect square: $(0.8y)^2$

$9x^2 - 0.64y^2 = (3x)^2 - (0.8y)^2$

$= (3x - 0.8y)(3x + 0.8y)$

**Your Turn**

Factor.

**a)** $3x^2 + 3x - 6$

**b)** $\frac{1}{2}x^2 - x - 4$

**c)** $0.49y^2 - 36k^2$
Example 2

Factor Polynomials of Quadratic Form

Factor each polynomial.

\( a) \ 12(x + 2)^2 + 24(x + 2) + 9 \)
\( b) \ 9(2t + 1)^2 - 4(s - 2)^2 \)

Solution

\( a) \ 12(x + 2)^2 + 24(x + 2) + 9 \)

Treat the term \( x + 2 \) as a single variable.
Substitute \( r = x + 2 \) into the quadratic expression and factor as usual.

\[
12(x + 2)^2 + 24(x + 2) + 9 \\
= 12r^2 + 24r + 9 \\
= 3(4r^2 + 8r + 3) \\
= 3(4r^2 + 2r + 6r + 3) \\
= 3[(4r^2 + 2r) + (6r + 3)] \\
= 3[2r(2r + 1) + 3(2r + 1)] \\
= 3(2r + 1)(2r + 3) \\
= 3(2(x + 2) + 1)[2(x + 2) + 3] \\
= 3(2x + 4 + 1)(2x + 4 + 3) \\
= 3(2x + 5)(2x + 7)
\]

The expression \( 12(x + 2)^2 + 24(x + 2) + 9 \) in factored form is \( 3(2x + 5)(2x + 7) \).

\( b) \ 9(2t + 1)^2 - 4(s - 2)^2 \)

Each term of the polynomial is a perfect square.
Therefore, this is a difference of squares of the form
\( P^2 - Q^2 = (P - Q)(P + Q) \) where \( P \) represents \( 3(2t + 1) \) and \( Q \) represents \( 2(s - 2) \).

Use the pattern for factoring a difference of squares.

\[
9(2t + 1)^2 - 4(s - 2)^2 \\
= [3(2t + 1) - 2(s - 2)][3(2t + 1) + 2(s - 2)] \\
= (6t + 3 - 2s + 4)(6t + 3 + 2s - 4) \\
= (6t - 2s + 7)(6t + 2s - 1)
\]

The expression \( 9(2t + 1)^2 - 4(s - 2)^2 \) in factored form is \( (6t - 2s + 7)(6t + 2s - 1) \).

Your Turn

Factor each polynomial.

\( a) \ -2(n + 3)^2 + 12(n + 3) + 14 \)
\( b) \ 4(x - 2)^2 - 0.25(y - 4)^2 \)
Solving Quadratic Equations by Factoring

Some quadratic equations that have real-number solutions can be factored easily.

The zero product property states that if the product of two real numbers is zero, then one or both of the numbers must be zero. This means that if \( de = 0 \), then at least one of \( d \) and \( e \) is 0.

The roots of a quadratic equation occur when the product of the factors is equal to zero. To solve a quadratic equation of the form \( ax^2 + bx + c = 0 \), \( a \neq 0 \), factor the expression and then set either factor equal to zero. The solutions are the roots of the equation.

For example, rewrite the quadratic equation \( 3x^2 - 2x - 5 = 0 \) in factored form.

\[
3x^2 - 2x - 5 = 0 \\
(3x - 5)(x + 1) = 0 \\
3x - 5 = 0 \quad \text{or} \quad x + 1 = 0 \\
x = \frac{5}{3} \quad x = -1
\]

The roots are \( \frac{5}{3} \) and \(-1\).

Example 3

Solve Quadratic Equations by Factoring

Determine the roots of each quadratic equation. Verify your solutions.

\( a) \quad x^2 + 6x + 9 = 0 \) \quad \( b) \quad x^2 + 4x - 21 = 0 \) \quad \( c) \quad 2x^2 - 9x - 5 = 0 \)

Solution

\( a) \quad \) To solve \( x^2 + 6x + 9 = 0 \), determine the factors and then solve for \( x \).

\[
x^2 + 6x + 9 = 0 \\
(x + 3)(x + 3) = 0
\]

\( x + 3 = 0 \quad \text{or} \quad x + 3 = 0 \)

\( x = -3 \quad x = -3 \)

For the quadratic equation to equal 0, one of the factors must equal 0.

This equation has two equal real roots. Since both roots are equal, the roots may be viewed as one distinct real root. Check by substituting the solution into the original quadratic equation.

For \( x = -3 \):

Left Side \hspace{1cm} Right Side
\[
x^2 + 6x + 9 = 0 \\
= (-3)^2 + 6(-3) + 9 \\
= 9 - 18 + 9 \\
= 0
\]

Left Side = Right Side

The solution is correct. The roots of the equation are \(-3\) and \(-3\), or just \(-3\).
b) To solve \( x^2 + 4x - 21 = 0 \), first determine the factors, and then solve for \( x \).

\[
\begin{align*}
x^2 + 4x - 21 &= 0 \\
(x - 3)(x + 7) &= 0
\end{align*}
\]

Two integers with a product of \(-21\) and a sum of \(4\) are \(-3\) and \(7\).

Set each factor equal to zero and solve for \( x \).

\[
\begin{align*}
x - 3 &= 0 & \text{or} & & x + 7 &= 0 \\
& \quad \Rightarrow x = 3 & & \quad \Rightarrow x = -7
\end{align*}
\]

The equation has two distinct real roots. Check by substituting each solution into the original quadratic equation.

For \( x = 3 \):

<table>
<thead>
<tr>
<th>Left Side</th>
<th>Right Side</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 + 4x - 21 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( = 3^2 + 4(3) - 21 )</td>
<td>( = 9 + 12 - 21 )</td>
</tr>
<tr>
<td>( = 9 - 21 )</td>
<td>( = 0 )</td>
</tr>
</tbody>
</table>

Left Side = Right Side

For \( x = -7 \):

<table>
<thead>
<tr>
<th>Left Side</th>
<th>Right Side</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 + 4x - 21 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( = (-7)^2 + 4(-7) - 21 )</td>
<td>( = 49 - 28 - 21 )</td>
</tr>
<tr>
<td>( = 21 )</td>
<td>( = 0 )</td>
</tr>
</tbody>
</table>

Left Side = Right Side

Both solutions are correct. The roots of the quadratic equation are 3 and \(-7\).

c) To solve \( 2x^2 - 9x - 5 = 0 \), first determine the factors, and then solve for \( x \).

**Method 1: Factor by Inspection**

\( 2x^2 \) is the product of the first terms, and \(-5\) is the product of the second terms.

\[
2x^2 - 9x - 5 = (2x + \_)(x + \_)
\]

The last term, \(-5\), is negative. So, one factor of \(-5\) must be negative. Try factor pairs of \(-5\) until the sum of the products of the outer and inner terms is \(-9x\).

<table>
<thead>
<tr>
<th>Factors of (-5)</th>
<th>Product</th>
<th>Middle Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-5, 1)</td>
<td>( (2x + 5)(x + 1) = 2x^2 + 2x - 5x - 5 )</td>
<td>(-3x) is not the correct middle term.</td>
</tr>
<tr>
<td>(1, -5)</td>
<td>( (2x + 1)(x - 5) = 2x^2 - 10x + 1x - 5 )</td>
<td>Correct.</td>
</tr>
</tbody>
</table>

Therefore, \( 2x^2 - 9x - 5 = (2x + 1)(x - 5) \).

\[
\begin{align*}
2x^2 - 9x - 5 &= 0 \\
(2x + 1)(x - 5) &= 0
\end{align*}
\]

Set each factor equal to zero and solve for \( x \).

\[
\begin{align*}
2x + 1 &= 0 & \text{or} & & x - 5 &= 0 \\
& \quad \Rightarrow 2x = -1 & & \quad \Rightarrow x = 5 \\
& \quad \Rightarrow x = -\frac{1}{2}
\end{align*}
\]

The roots are \(-\frac{1}{2}\) and 5.
Method 2: Factor by Grouping
Find two integers with a product of \((2)(-5) = -10\) and a sum of \(-9\).

<table>
<thead>
<tr>
<th>Factors of (-10)</th>
<th>Product</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, -10</td>
<td>-10</td>
<td>-9</td>
</tr>
<tr>
<td>2, -5</td>
<td>-10</td>
<td>-3</td>
</tr>
<tr>
<td>5, -2</td>
<td>-10</td>
<td>3</td>
</tr>
<tr>
<td>10, 1</td>
<td>-10</td>
<td>9</td>
</tr>
</tbody>
</table>

Write \(-9x\) as \(x - 10x\). Then, factor by grouping.
\[
2x^2 - 9x - 5 = 0
\]
\[
2x^2 + x - 10x - 5 = 0
\]
\[
(2x^2 + x) + (-10x - 5) = 0
\]
\[
x(2x + 1) - 5(2x + 1) = 0
\]
\[
(2x + 1)(x - 5) = 0
\]

Set each factor equal to zero and solve for \(x\).
\[2x + 1 = 0 \quad \text{or} \quad x - 5 = 0\]
\[
2x = -1 \quad \quad \quad x = 5
\]
\[
x = \frac{-1}{2}
\]

The roots are \(-\frac{1}{2}\) and 5.

Check for both Methods 1 and 2:
The equation has two distinct real roots. Check by substituting each root into the original quadratic equation.

For \(x = -\frac{1}{2}\):

<table>
<thead>
<tr>
<th>Left Side</th>
<th>Right Side</th>
</tr>
</thead>
</table>
| \[
2x^2 - 9x - 5
\]
| \[
= 2\left(\frac{-1}{2}\right)^2 - 9\left(\frac{-1}{2}\right) - 5
\]
| \[
= 2\left(\frac{1}{4}\right) + \frac{9}{2} - 5
\]
| \[
= \frac{1}{2} + \frac{9}{2} - 10
\]
| \[
= 0
\]

Left Side = Right Side

Both solutions are correct.
The roots of the quadratic equation are \(-\frac{1}{2}\) and 5.

Your Turn
Determine the roots of each quadratic equation.

a) \(x^2 - 10x + 25 = 0\)
b) \(x^2 - 16 = 0\)
c) \(3x^2 - 2x - 8 = 0\)
Example 4
Apply Quadratic Equations

Dock jumping is an exciting dog event in which dogs compete for the longest jumping distance from a dock into a body of water. The path of a Jack Russell terrier on a particular jump can be approximated by the quadratic function $h(d) = -\frac{3}{10}d^2 + \frac{11}{10}d + 2$, where $h$ is the height above the surface of the water and $d$ is the horizontal distance the dog travels from the base of the dock, both in feet. All measurements are taken from the base of the dog’s tail. Determine the horizontal distance of the jump.

Solution

When the dog lands in the water, the dog’s height above the surface is 0 m. To solve this problem, determine the roots of the quadratic equation $-\frac{3}{10}d^2 + \frac{11}{10}d + 2 = 0$.

$$-\frac{3}{10}d^2 + \frac{11}{10}d + 2 = 0$$

$$-\frac{1}{10}(3d^2 - 11d - 20) = 0$$

$$-\frac{1}{10}(3d + 4)(d - 5) = 0$$

$$3d + 4 = 0 \quad \text{or} \quad d - 5 = 0$$

$$3d = -4 \quad \text{or} \quad d = 5$$

$$d = -\frac{4}{3}$$

Factor out the common factor of $-\frac{1}{10}$.

Solve for $d$ to determine the roots of the equation.

Why does the factor $-\frac{1}{10}$ neither result in a root nor affect the other roots of the equation?
Since $d$ represents the horizontal distance of the dog from the base of the dock, it cannot be negative.

So, reject the root $-\frac{4}{3}$.

Check the solution by substituting $d = 5$ into the original quadratic equation.

For $d = 5$:

Left Side \quad Right Side

\[
-\frac{3}{10}d^2 + \frac{11}{10}d + 2 = 0
\]

\[= -\frac{3}{10}(5)^2 + \frac{11}{10}(5) + 2\]

\[= -\frac{15}{2} + \frac{11}{2} + \frac{4}{2}\]

\[= 0\]

Left Side $=$ Right Side

The solution is correct.
The dog travels a horizontal distance of 5 ft.

**Your Turn**

A waterslide ends with the slider dropping into a deep pool of water. The path of the slider after leaving the lower end of the slide can be approximated by the quadratic function

\[h(d) = -\frac{1}{6}d^2 - \frac{1}{6}d + 2,\] where $h$ is the height above the surface of the pool and $d$ is the horizontal distance the slider travels from the lower end of the slide, both in feet. What is the horizontal distance the slider travels before dropping into the pool after leaving the lower end of the slide?
Example 5
Write and Solve a Quadratic Equation

The length of an outdoor lacrosse field is 10 m less than twice the width. The area of the field is 6600 m². Determine the dimensions of an outdoor lacrosse field.

Solution
Let $w$ represent the width of the field. Then, the length of the field is $2w - 10$.

Use the area formula.

$$A = lw$$

$$6600 = (2w - 10)(w)$$

$$6600 = 2w^2 - 10w$$

$$0 = 2w^2 - 10w - 6600$$

$$0 = 2(w^2 - 5w - 3300)$$

$$0 = w^2 - 5w - 3300$$

$$0 = (w - 60)(w + 55)$$

$$w - 60 = 0 \quad \text{or} \quad w + 55 = 0$$

$$w = 60 \quad w = -55$$

Since the width of the field cannot be negative, $w = -55$ is rejected. The width of the field is 60 m. The length of the field is $2(60) - 10$ or 110 m.

Check:
The area of the field is $(60)(110)$ or 6600 m².

Your Turn
The area of a rectangular Ping-Pong table is 45 ft². The length is 4 ft more than the width. What are the dimensions of the table?
Key Ideas

- You can solve some quadratic equations by factoring.
- If two factors of a quadratic equation have a product of zero, then by the zero product property one of the factors must be equal to zero.
- To solve a quadratic equation by factoring, first write the equation in the form \( ax^2 + bx + c = 0 \), and then factor the left side. Next, set each factor equal to zero, and solve for the unknown.

For example,
\[
\begin{align*}
    x^2 + 8x &= -12 \\
    x^2 + 8x + 12 &= 0 \\
    (x + 2)(x + 6) &= 0 \\
    x + 2 &= 0 \quad \text{or} \quad x + 6 &= 0 \\
    x &= -2 \quad \quad x = -6
\end{align*}
\]
- The solutions to a quadratic equation are called the roots of the equation.
- You can factor polynomials in quadratic form.
  - Factor trinomials of the form \( aP^2 + bP + c \), where \( a \neq 0 \) and \( P \) is any expression, by replacing the expression for \( P \) with a single variable. Then substitute the expression for \( P \) back into the factored expression. Simplify the final factors, if possible.

For example, factor \( 2(x + 3)^2 - 11(x + 3) + 15 \) by letting \( r = x + 3 \).
\[
\begin{align*}
    2(x + 3)^2 - 11(x + 3) + 15 &= 2r^2 - 11r + 15 \\
    &= 2r^2 - 5r - 6r + 15 \\
    &= (2r^2 - 5r) + (-6r + 15) \\
    &= r(2r - 5) - 3(2r - 5) \\
    &= (2r - 5)(r - 3) \\
    &= [2(x + 3) - 5][x + 3] - 3 \\
    &= (2x + 1)(x) \\
    &= x(2x + 1)
\end{align*}
\]
- Factor a difference of squares, \( P^2 - Q^2 \), where \( P \) and \( Q \) are any expressions, as \( [P - Q][P + Q] \).

Check Your Understanding

Practise

1. Factor completely.
   a) \( x^2 + 7x + 10 \)
   b) \( 5z^2 + 40z + 60 \)
   c) \( 0.2d^2 - 2.2d + 5.6 \)

2. Factor completely.
   a) \( 3y^2 + 4y - 7 \)
   b) \( 8k^2 - 6k - 5 \)
   c) \( 0.4m^2 + 0.6m - 1.8 \)
3. Factor completely.
   a) \(x^2 + x - 20\)
   b) \(x^2 - 12x + 36\)
   c) \(\frac{1}{4}x^2 + 2x + 3\)
   d) \(2x^2 + 12x + 18\)

4. Factor each expression.
   a) \(4y^2 - 9x^2\)
   b) \(0.36p^2 - 0.49q^2\)
   c) \(\frac{1}{4}s^2 - \frac{9}{25}t^2\)
   d) \(0.16t^2 - 16s^2\)

5. Factor each expression.
   a) \((x + 2)^2 - (x + 2) - 42\)
   b) \(6(x^2 - 4x + 4)^2 + (x^2 - 4x + 4) - 1\)
   c) \((4j - 2)^2 - (2 + 4j)^2\)

6. What are the factors of each expression?
   a) \(4(5b^2 - 3)^2 + 10(5b - 3) - 6\)
   b) \(16(x^2 + 1)^2 - 4(2x)^2\)
   c) \(-\frac{1}{4}(2x)^2 + 25(2y)^2\)

7. Solve each factored equation.
   a) \((x + 3)(x + 4) = 0\)
   b) \((x - 2)(x + \frac{1}{2}) = 0\)
   c) \((x + 7)(x - 8) = 0\)
   d) \(x(x + 5) = 0\)
   e) \((3x + 1)(5x - 4) = 0\)
   f) \(2(x - 4)(7 - 2x) = 0\)

8. Solve each quadratic equation by factoring. Check your answers.
   a) \(10n^2 - 40 = 0\)
   b) \(\frac{1}{4}x^2 + \frac{5}{4}x + 1 = 0\)
   c) \(3w^2 + 28w + 9 = 0\)
   d) \(8y^2 - 22y + 15 = 0\)
   e) \(d^2 + \frac{5}{2}d + \frac{3}{2} = 0\)
   f) \(4x^2 - 12x + 9 = 0\)

9. Determine the roots of each quadratic equation. Verify your answers.
   a) \(k^2 - 5k = 0\)
   b) \(9x^2 = x + 8\)
   c) \(\frac{8}{3}t + 5 = -\frac{1}{3}t^2\)
   d) \(\frac{25}{49}x^2 - 9 = 0\)
   e) \(2s^2 - 4s = 70\)
   f) \(4q^2 - 28q = -49\)

10. Solve each equation.
    a) \(42 = x^2 - x\)
    b) \(g^2 = 30 - 7g\)
    c) \(y^2 + 4y = 21\)
    d) \(3 = 6p^2 - 7p\)
    e) \(3x^2 + 9x = 30\)
    f) \(2z^2 = 3 - 5z\)

Apply

11. A rectangle has dimensions \(x + 10\) and \(2x - 3\), where \(x\) is in centimetres. The area of the rectangle is 54 cm\(^2\).

   \[
   \begin{array}{c}
   2x - 3 \\
   x + 10
   \end{array}
   \]

   a) What equation could you use to determine the value of \(x\)?

   b) What is the value of \(x\)?

12. An osprey, a fish-eating bird of prey, dives toward the water to catch a salmon. The height, \(h\), in metres, of the osprey above the water \(t\) seconds after it begins its dive can be approximated by the function \(h(t) = 5t^2 - 30t + 45\).

   a) Determine the time it takes for the osprey to reach a height of 20 m.

13. A flare is launched from a boat. The height, $h$, in metres, of the flare above the water is approximately modelled by the function $h(t) = 150t - 5t^2$, where $t$ is the number of seconds after the flare is launched.

a) What equation could you use to determine the time it takes for the flare to return to the water?

b) How many seconds will it take for the flare to return to the water?

16. Ted popped a baseball straight up with an initial upward velocity of 48 ft/s. The height, $h$, in feet, of the ball above the ground is modelled by the function $h(t) = 3 + 48t - 16t^2$. How long was the ball in the air if the catcher catches the ball 3 ft above the ground? Is your answer reasonable in this situation? Explain.

Did You Know?

Many Canadians have made a positive impact on Major League Baseball. Players such as Larry Walker of Maple Ridge, British Columbia, Jason Bay of Trail, British Columbia, and Justin Morneau of Westminster, British Columbia have had very successful careers in baseball’s highest league.

17. A rectangle with area of 35 cm$^2$ is formed by cutting off strips of equal width from a rectangular piece of paper.

a) What is the width of each strip?

b) What are the dimensions of the new rectangle?

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4.2 Factoring Quadratic Equations • MHR 231
18. Without factoring, state if the binomial is a factor of the trinomial. Explain why or why not.
   a) \(x^2 - 5x - 36, \ x - 5\)
   b) \(x^2 - 2x - 15, \ x + 3\)
   c) \(6x^2 + 11x + 4, \ 4x + 1\)
   d) \(4x^2 + 4x - 3, \ 2x - 1\)

19. Solve each equation.
   a) \(x(2x - 3) - 2(3 + 2x) = -4(x + 1)\)
   b) \(3(x - 2)(x + 1) - 4 = 2(x - 1)^3\)

20. The hypotenuse of a right triangle measures 29 cm. One leg is 1 cm shorter than the other. What are the lengths of the legs?

21. A field is in the shape of a right triangle. The fence around the perimeter of the field measures 40 m. If the length of the hypotenuse is 17 m, find the length of the other two sides.

22. The width of the top of a notebook computer is 7 cm less than the length. The surface area of the top of the notebook is 690 cm².
   a) Write an equation to represent the surface area of the top of the notebook computer.
   b) What are the dimensions of the top of the computer?

23. Stephan plans to build a uniform walkway around a rectangular flower bed that is 20 m by 40 m. There is enough material to make a walkway that has a total area of 700 m². What is the width of the walkway?

24. An 18-m-tall tree is broken during a severe storm, as shown. The distance from the base of the trunk to the point where the tip touches the ground is 12 m. At what height did the tree break?

25. The pressure difference, \(P\), in newtons per square metre, above and below an airplane wing is described by the formula 
\[ P = \left(\frac{1}{2}d\right)(v_1)^2 - \left(\frac{1}{2}d\right)(v_2)^2, \] where \(d\) is the density of the air, in kilograms per cubic metre; \(v_1\) is the velocity, in metres per second, of the air passing above; and \(v_2\) is the velocity, in metres per second, of the air passing below. Write this formula in factored form.

26. Carlos was asked to factor the trinomial \(6x^2 - 16x + 8\) completely. His work is shown below.
   
   **Carlos's solution:**
   \[ 6x^2 - 16x + 8 \]
   \[ = 6x^2 - 12x - 4x + 8 \]
   \[ = 6x(x - 2) - 4(x - 2) \]
   \[ = (x - 2)(6x - 4) \]
   Is Carlos correct? Explain.

27. Factor each expression.
   a) \(3(2z + 3)^2 - 9(2z + 3) - 30\)
   b) \(16(m^2 - 4)^3 - 4(3m)^3\)
   c) \(\frac{1}{9}y^2 - \frac{1}{3}yx + \frac{1}{4}x^2\)
   d) \(-2\left(w + \frac{2}{3}\right)^2 + 7\left(3w - \frac{1}{3}\right)^2\)

**Extend**

28. A square has an area of \((9x^2 + 30xy + 25y^2)\) square centimetres. What is an expression for the perimeter of the square?
29. Angela opened a surf shop in Tofino, British Columbia. Her accountant models her profit, \( P \), in dollars, with the function \( P(t) = 1125(t - 1)^2 - 10 \ 125 \), where \( t \) is the number of years of operation. Use graphing or factoring to determine how long it will take for the shop to start making a profit.

Create Connections

30. Write a quadratic equation in standard form with the given root(s).
   a) \(-3\) and \(3\)
   b) \(2\)
   c) \(\frac{2}{3}\) and \(4\)
   d) \(\frac{3}{5}\) and \(-\frac{1}{2}\)

31. Create an example of a quadratic equation that cannot be solved by factoring. Explain why it cannot be factored. Show the graph of the corresponding quadratic function and show where the roots are located.

32. You can use the difference of squares pattern to perform certain mental math shortcuts. For example,
   \[81 - 36 = (9 - 6)(9 + 6) = (3)(15) = 45\]
   a) Explain how this strategy works. When can you use it?
   b) Create two examples to illustrate the strategy.

Did You Know?

Pete Devries was the first Canadian to win an international surfing competition. In 2009, he outperformed over 110 world-class surfers to win the O'Neill Cold Water Classic Canada held in Tofino, British Columbia.

Avalanche Safety

- Experts use avalanche control all over the world above highways, ski resorts, railroads, mining operations, and utility companies, and anywhere else that may be threatened by avalanches.
- Avalanche control is the intentional triggering of avalanches. People are cleared away to a safe distance, then experts produce more frequent, but smaller, avalanches at controlled times.
- Because avalanches tend to occur in the same zones and under certain conditions, avalanche experts can predict when avalanches are likely to occur.
- Charges are delivered by launchers, thrown out of helicopters, or delivered above the avalanche starting zones by an avalanche control expert on skis.
- What precautions would avalanche control experts need to take to ensure public safety?