Graphical Solutions of Quadratic Equations

Focus on...
- describing the relationships between the roots of a quadratic equation, the zeros of the corresponding quadratic function, and the x-intercepts of the graph of the quadratic function
- solving quadratic equations by graphing the corresponding quadratic function

Water fountains are usually designed to give a specific visual effect. For example, the water fountain shown consists of individual jets of water that arch up in the shape of a parabola. Notice how the jets of water are designed to land precisely on the underwater spotlights.

How can you design a water fountain to do this? Where must you place the underwater lights so the jets of water land on them? What are some of the factors to consider when designing a water fountain? How do these factors affect the shape of the water fountain?

Investigate Solving Quadratic Equations by Graphing

Materials
- grid paper or graphing technology

1. Each water fountain jet creates a parabolic stream of water. You can represent this curve by the quadratic function \( h(x) = -6(x - 1)^2 + 6 \), where \( h \) is the height of the jet of water and \( x \) is the horizontal distance of the jet of water from the nozzle, both in metres.
a) Graph the quadratic function \( h(x) = -6(x - 1)^2 + 6 \).

b) How far from the nozzle should the underwater lights be placed? Explain your reasoning.

2. You can control the height and horizontal distance of the jet of water by changing the water pressure. Suppose that the quadratic function \( h(x) = -x^2 + 12x \) models the path of a jet of water at maximum pressure. The quadratic function \( h(x) = -3x^2 + 12x \) models the path of the same jet of water at a lower pressure.

a) Graph these two functions on the same set of axes as in step 1.

b) Describe what you notice about the \( x \)-intercepts and height of the two graphs compared to the graph in step 1.

c) Why do you think the \( x \)-intercepts of the graph are called the zeros of the function?

Reflect and Respond

3. a) If the water pressure in the fountain must remain constant, how else could you control the path of the jets of water?

b) Could two jets of water at constant water pressure with different parabolic paths land on the same spot? Explain your reasoning.

Did You Know?

The Dubai Fountain at the Burj Khalifa in Dubai is the largest in the world. It can shoot about 22 000 gal of water about 500 ft into the air and features over 6600 lights and 25 colour projectors.
quadratic equation
- a second-degree equation with standard form $ax^2 + bx + c = 0$, where $a \neq 0$
- for example, $2x^2 + 12x + 16 = 0$

root(s) of an equation
- the solution(s) to an equation

zero(s) of a function
- the value(s) of $x$ for which $f(x) = 0$
- related to the x-intercept(s) of the graph of a function, $f(x)$

You can solve a quadratic equation of the form $ax^2 + bx + c = 0$ by graphing the corresponding quadratic function, $f(x) = ax^2 + bx + c$. The solutions to a quadratic equation are called the roots of the equation. You can find the roots of a quadratic equation by determining the x-intercepts of the graph, or the zeros of the corresponding quadratic function.

For example, you can solve the quadratic equation $2x^2 + 2x - 12 = 0$ by graphing the corresponding quadratic function, $f(x) = 2x^2 + 2x - 12$. The graph shows that the x-intercepts occur at $(-3, 0)$ and $(2, 0)$ and have values of $-3$ and $2$. The zeros of the function occur when $f(x) = 0$. So, the zeros of the equation are $-3$ and $2$.

Therefore, the roots of the equation are $-3$ and $2$.

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Example 1

Quadratic Equations With One Real Root

What are the roots of the equation $-x^2 + 8x - 16 = 0$?

Solution

To solve the equation, graph the corresponding quadratic function, $f(x) = -x^2 + 8x - 16$, and determine the x-intercepts.

Method 1: Use Paper and Pencil

Create a table of values. Plot the coordinate pairs and use them to sketch the graph of the function.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-36</td>
</tr>
<tr>
<td>-1</td>
<td>-25</td>
</tr>
<tr>
<td>0</td>
<td>-16</td>
</tr>
<tr>
<td>1</td>
<td>-9</td>
</tr>
<tr>
<td>2</td>
<td>-4</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>-1</td>
</tr>
<tr>
<td>6</td>
<td>-4</td>
</tr>
<tr>
<td>7</td>
<td>-9</td>
</tr>
<tr>
<td>8</td>
<td>-16</td>
</tr>
<tr>
<td>9</td>
<td>-25</td>
</tr>
<tr>
<td>10</td>
<td>-36</td>
</tr>
</tbody>
</table>

Why were these values of $x$ chosen?

How do you know that there is only one root for this quadratic equation?
The graph meets the x-axis at the point (4, 0), the vertex of the corresponding quadratic function. The x-intercept of the graph occurs at (4, 0) and has a value of 4. The zero of the function is 4. Therefore, the root of the equation is 4.

**Method 2: Use a Spreadsheet**

In a spreadsheet, enter the table of values shown. Then, use the spreadsheet’s graphing features.

The x-intercept of the graph occurs at (4, 0) and has a value of 4.

The zero of the function is 4.

Therefore, the root of the equation is 4.

**Method 3: Use a Graphing Calculator**

Graph the function using a graphing calculator. Then, use the trace or zero function to identify the x-intercept.

The x-intercept of the graph occurs at (4, 0) and has a value of 4.

The zero of the function is 4.

Therefore, the root of the equation is 4.

Check for Methods 1, 2, and 3:
Substitute $x = 4$ into the equation $-x^2 + 8x - 16 = 0$.

<table>
<thead>
<tr>
<th>Left Side</th>
<th>Right Side</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-x^2 + 8x - 16$</td>
<td>0</td>
</tr>
<tr>
<td>$= -(4)^2 + 0(4) - 16$</td>
<td>$= -16 + 32 - 16$</td>
</tr>
<tr>
<td>$= 0$</td>
<td>$= 0$</td>
</tr>
</tbody>
</table>

Left Side = Right Side

The solution is correct.

**Your Turn**

Determine the roots of the quadratic equation $x^2 - 6x + 9 = 0$. 

4.1 Graphical Solutions of Quadratic Equations • MHR 209
Example 2

Quadratic Equations With Two Distinct Real Roots

The manager of Jasmine's Fine Fashions is investigating the effect that raising or lowering dress prices has on the daily revenue from dress sales. The function \( R(x) = 100 + 15x - x^2 \) gives the store's revenue \( R \), in dollars, from dress sales, where \( x \) is the price change, in dollars. What price changes will result in no revenue?

Solution

When there is no revenue, \( R(x) = 0 \). To determine the price changes that result in no revenue, solve the quadratic equation \( 0 = 100 + 15x - x^2 \).

Graph the corresponding revenue function. On the graph, the \( x \)-intercepts will correspond to the price changes that result in no revenue. What do the values of \( x \) that are not the \( x \)-intercepts represent?

Method 1: Use Paper and Pencil

Create a table of values. Plot the coordinate pairs and use them to sketch the graph of the function.

Why do the values of \( x \) in the table begin with negative values?

<table>
<thead>
<tr>
<th>Price Change, ( x )</th>
<th>Revenue, ( R(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-10</td>
<td>-150</td>
</tr>
<tr>
<td>-8</td>
<td>-84</td>
</tr>
<tr>
<td>-6</td>
<td>-26</td>
</tr>
<tr>
<td>-4</td>
<td>24</td>
</tr>
<tr>
<td>-2</td>
<td>66</td>
</tr>
<tr>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>2</td>
<td>126</td>
</tr>
<tr>
<td>4</td>
<td>144</td>
</tr>
<tr>
<td>6</td>
<td>154</td>
</tr>
<tr>
<td>8</td>
<td>156</td>
</tr>
<tr>
<td>10</td>
<td>150</td>
</tr>
<tr>
<td>12</td>
<td>136</td>
</tr>
<tr>
<td>14</td>
<td>114</td>
</tr>
<tr>
<td>16</td>
<td>84</td>
</tr>
<tr>
<td>18</td>
<td>46</td>
</tr>
<tr>
<td>20</td>
<td>0</td>
</tr>
<tr>
<td>22</td>
<td>-54</td>
</tr>
</tbody>
</table>

The graph appears to cross the \( x \)-axis at the points \((-5, 0)\) and \((20, 0)\). The \( x \)-intercepts of the graph, or zeros of the function, are \(-5\) and \(20\).

Therefore, the roots of the equation are \(-5\) and \(20\). Why do the roots of the equation result in no revenue?
Method 2: Use a Spreadsheet
In a spreadsheet, enter the table of values shown. Then, use the spreadsheet’s graphing features.

The graph crosses the x-axis at the points (−5, 0) and (20, 0). The x-intercepts of the graph, or zeros of the function, are −5 and 20. Therefore, the roots of the equation are −5 and 20.

Method 3: Use a Graphing Calculator
Graph the revenue function using a graphing calculator. Adjust the window settings of the graph until you see the vertex of the parabola and the x-intercepts. Use the trace or zero function to identify the x-intercepts of the graph.

The graph crosses the x-axis at the points (−5, 0) and (20, 0). The x-intercepts of the graph, or zeros of the function, are −5 and 20. Therefore, the roots of the equation are −5 and 20.

Check for Methods 1, 2, and 3:
Substitute the values \(x = -5\) and \(x = 20\) into the equation
\[0 = 100 + 15x - x^2.\]

<table>
<thead>
<tr>
<th>Left Side</th>
<th>Right Side</th>
<th>Left Side</th>
<th>Right Side</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>100 + 15(-5) - (-5)^2</td>
<td>0</td>
<td>100 + 15(20) - (20)^2</td>
</tr>
<tr>
<td></td>
<td>= 100 - 75 - 25</td>
<td></td>
<td>= 100 + 300 - 400</td>
</tr>
<tr>
<td></td>
<td>= 0</td>
<td></td>
<td>= 0</td>
</tr>
</tbody>
</table>

Left Side = Right Side

Both solutions are correct. A dress price increase of $20 or a decrease of $5 will result in no revenue from dress sales.
Your Turn
The manager at Suzie’s Fashion Store has determined that the function
\[ R(x) = 600 - 6x^2 \]
models the expected weekly revenue, \( R \), in dollars, from sweatshirts as the price changes, where \( x \) is the change in price, in dollars.
What price increase or decrease will result in no revenue?

Example 3
Quadratic Equations With No Real Roots
Solve \( 2x^2 + x = -2 \) by graphing.

Solution
Rewrite the equation in the form \( ax^2 + bx + c = 0 \).
\[ 2x^2 + x + 2 = 0 \]
Why do you rewrite the equation in the form \( ax^2 + bx + c = 0 \)?
Graph the corresponding quadratic function \( f(x) = 2x^2 + x + 2 \).

The graph does not intersect the x-axis.
There are no zeros for this function.
Therefore, the quadratic equation has no real roots.

Your Turn
Solve \( 3m^2 - m = -2 \) by graphing.
Key Ideas

- One approach to solving a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$, is to graph the corresponding quadratic function, $f(x) = ax^2 + bx + c$. Then, determine the x-intercepts of the graph.

- The x-intercepts of the graph, or the zeros of the quadratic function, correspond to the solutions, or roots, of the quadratic equation.

  For example, you can solve $x^2 - 5x + 6 = 0$ by graphing the corresponding function, $f(x) = x^2 - 5x + 6$, and determining the x-intercepts.

  The x-intercepts of the graph and the zeros of the function are 2 and 3. So, the roots of the equation are 2 and 3.

  Check:
  Substitute the values $x = 2$ and $x = 3$ into the equation $x^2 - 5x + 6 = 0$.

  \[
  \begin{align*}
  \text{Left Side} & \quad \text{Right Side} \\
  x^2 - 5x + 6 & \quad 0 \\
  = (2)^2 - 5(2) + 6 & \quad = (3)^2 - 5(3) + 6 \\
  = 4 - 10 + 6 & \quad = 9 - 15 + 6 \\
  = 0 & \quad = 0 \\
  \text{Left Side} = \text{Right Side} & \quad \text{Left Side} = \text{Right Side}
  \end{align*}
  \]

  Both solutions are correct.

- The graph of a quadratic function can have zero, one, or two real x-intercepts. Therefore, the quadratic function has zero, one, or two real zeros, and correspondingly the quadratic equation has zero, one, or two real roots.

![Graphs showing different cases of x-intercepts](image_url)
Practise

1. How many x-intercepts does each quadratic function graph have?

3. Solve each equation by graphing the corresponding function.
   
   a) \( 0 = x^2 - 5x - 24 \)
   
   b) \( 0 = -2r^2 - 6r \)
   
   c) \( h^2 + 2h + 5 = 0 \)
   
   d) \( 5x^2 - 5x = 30 \)
   
   e) \( -2x^2 + 4x = 4 \)
   
   f) \( 0 = t^2 + 4t + 10 \)

4. What are the roots of each quadratic equation? Where integral roots cannot be found, estimate the roots to the nearest tenth.

   a) \( n^2 - 10 = 0 \)
   
   b) \( 0 = 3x^2 + 9x - 12 \)
   
   c) \( 0 = -w^2 + 4w - 3 \)
   
   d) \( 0 = 2d^2 + 20d - 32 \)
   
   e) \( 0 = v^2 + 6v + 6 \)
   
   f) \( m^2 - 10m = -21 \)

Apply

5. In a Canadian Football League game, the path of the football at one particular kick-off can be modelled using the function \( h(d) = -0.02d^2 + 2.6d - 66.5 \), where \( h \) is the height of the ball and \( d \) is the horizontal distance from the kicking team's goal line, both in yards. A value of \( h(d) = 0 \) represents the height of the ball at ground level. What horizontal distance does the ball travel before it hits the ground?

6. Two numbers have a sum of 9 and a product of 20.

   a) What single-variable quadratic equation in the form \( ax^2 + bx + c = 0 \) can be used to represent the product of the two numbers?

   b) Determine the two numbers by graphing the corresponding quadratic function.
7. Two consecutive even integers have a product of 168.
   a) What single-variable quadratic equation in the form \( ax^2 + bx + c = 0 \) can be used to represent the product of the two numbers?
   b) Determine the two numbers by graphing the corresponding quadratic function.

8. The path of the stream of water coming out of a fire hose can be approximated using the function \( h(x) = -0.09x^2 + x + 1.2 \), where \( h \) is the height of the water stream and \( x \) is the horizontal distance from the firefighter holding the nozzle, both in metres.
   a) What does the equation \(-0.09x^2 + x + 1.2 = 0\) represent in this situation?
   b) At what maximum distance from the building could a firefighter stand and still reach the base of the fire with the water? Express your answer to the nearest tenth of a metre.
   c) What assumptions did you make when solving this problem?

9. The HSBC Celebration of Light is an annual pyro-musical fireworks competition that takes place over English Bay in Vancouver. The fireworks are set off from a barge so they land on the water. The path of a particular fireworks rocket is modelled by the function \( h(t) = -4.9(t - 3)^2 + 47 \), where \( h \) is the rocket's height above the water, in metres, at time, \( t \), in seconds.
   a) What does the equation \( 0 = -4.9(t - 3)^2 + 47 \) represent in this situation?
   b) The fireworks rocket stays lit until it hits the water. For how long is it lit, to the nearest tenth of a second?

10. A skateboarder jumps off a ledge at a skateboard park. His path is modelled by the function \( h(d) = -0.75d^2 + 0.9d + 1.5 \), where \( h \) is the height above ground and \( d \) is the horizontal distance the skateboarder travels from the ledge, both in metres.
   a) Write a quadratic equation to represent the situation when the skateboarder lands.
   b) At what distance from the base of the ledge will the skateboarder land? Express your answer to the nearest tenth of a metre.

11. Émilie Heymans is a three-time Canadian Olympic diving medallist. Suppose that for a dive off the 10-m tower, her height, \( h \), in metres, above the surface of the water is given by the function \( h(d) = -2d^2 + 3d + 10 \), where \( d \) is the horizontal distance from the end of the tower platform, in metres.
   a) Write a quadratic equation to represent the situation when Émilie enters the water.
   b) What is Émilie’s horizontal distance from the end of the tower platform when she enters the water? Express your answer to the nearest tenth of a metre.

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**Did You Know?**

Émilie Heymans, from Montréal, Québec, is only the fifth Canadian to win medals at three consecutive Olympic Games.
12. Matthew is investigating the old Borden Bridge, which spans the North Saskatchewan River about 50 km west of Saskatoon. The three parabolic arches of the bridge can be modelled using quadratic functions, where \( h \) is the height of the arch above the bridge deck and \( x \) is the horizontal distance of the bridge deck from the beginning of the first arch, both in metres.

First arch:
\[
h(x) = -0.01x^2 + 0.84x
\]
Second arch:
\[
h(x) = -0.01x^2 + 2.52x - 141.12
\]
Third arch:
\[
h(x) = -0.01x^2 + 4.2x - 423.36
\]
a) What are the zeros of each quadratic function?

b) What is the significance of the zeros in this situation?

c) What is the total span of the Borden Bridge?

14. The height of a circular arch is represented by \( 4h^2 - 8hr + s^2 = 0 \), where \( h \) is the height, \( r \) is the radius, and \( s \) is the span of the arch, both in feet.

a) How high must an arch be to have a span of 64 ft and a radius of 40 ft?

b) How would this equation change if all the measurements were in metres? Explain.

15. Two new hybrid vehicles accelerate at different rates. The Ultra Range's acceleration can be modelled by the function \( d(t) = 1.5t^2 \), while the Edison's can be modelled by the function \( d(t) = 5.4t^2 \), where \( d \) is the distance, in metres, and \( t \) is the time, in seconds. The Ultra Range starts the race at 0 s. At what time should the Edison start so that both cars are at the same point 5 s after the race starts? Express your answer to the nearest tenth of a second.

**Did You Know?**
A hybrid vehicle uses two or more distinct power sources. The most common hybrid uses a combination of an internal combustion engine and an electric motor. These are called hybrid electric vehicles or HEVs.

**Create Connections**
16. Suppose the value of a quadratic function is negative when \( x = 1 \) and positive when \( x = 2 \). Explain why it is reasonable to assume that the related equation has a root between 1 and 2.

17. The equation of the axis of symmetry of a quadratic function is \( x = 0 \) and one of the \( x \)-intercepts is \(-4\). What is the other \( x \)-intercept? Explain using a diagram.

18. The roots of the quadratic equation \( 0 = x^2 - 4x - 12 \) are 6 and \(-2\). How can you use the roots to determine the vertex of the graph of the corresponding function?